Consistent k-Clustering for General Metrics

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SODA 2021

Consistent Clustering

input model: stream of point insertions $\langle p_1, p_2, \ldots, p_n \rangle \rightarrow$ sets $P_1, P_2, \ldots$

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$$\underset{\substack{C_i \subset P_i \\ |C_i| = k}}{\arg\min} \sum_{p \in P_i} d(p, C_i) \qquad \forall i \in [n]$$

↑ general metric

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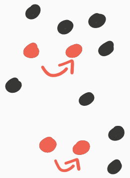

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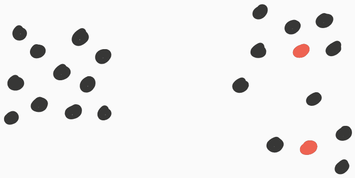

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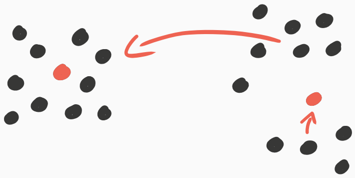

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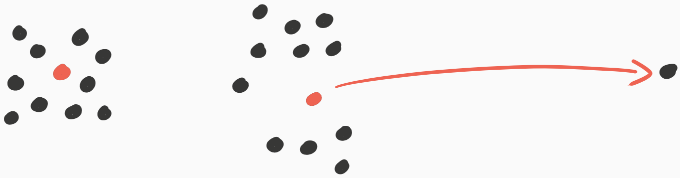

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consistency objective: approximate solution with as few center swaps as possible

$$\min \sum_{i \in [n]} |C_i \setminus C_{i-1}|$$

Consistent Clustering

Main Result

An insertion-only streaming algorithm that maintains an $O(1)$-approximate k-median solution and swaps at most $O(k \cdot \text{polylog}(n, \Delta))$ centers during the entire execution.

spread of metric

total number of points

number of centers

Lattanzi, Vassilvitskii, ICML'17; Guo et al., preprint '20; Cohen-Addad et al. NeuRIPS'19; Guo et al., APPROX'20

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Lower Bound [LV17]

$\Omega(k \log(n\Delta))$ swaps are neccessary even for offline setting.

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Related Work

Previous result [LV17]: $O(k^2 \log(n\Delta)^4)$

Deterministic with outliers [GKSX20]: $O(k^2 \log(n\Delta)^2)$

Dynamic consistent clustering [CHPSS19, GKLX20, ...]

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assumption: $\mathrm{OPT}_{\mathrm{guess}} \leq \mathrm{OPT}_k \leq c \cdot \mathrm{OPT}_{\mathrm{guess}}$

↖ recompute solution $O(\log(\Delta n))$ times from scratch

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[LV17]: $O(k \cdot m)$

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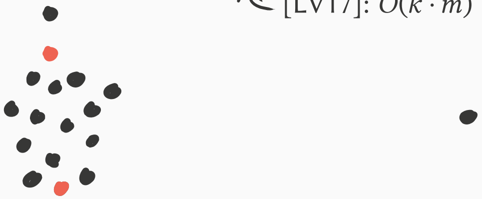

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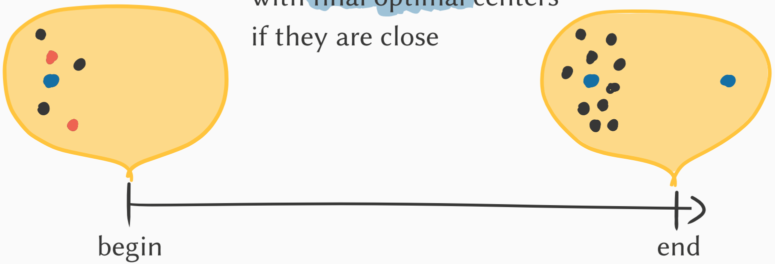
can only change O(1) centers per insertion!

Algorithm

Algorithm

pair maintained centers
with final optimal centers
if they are close



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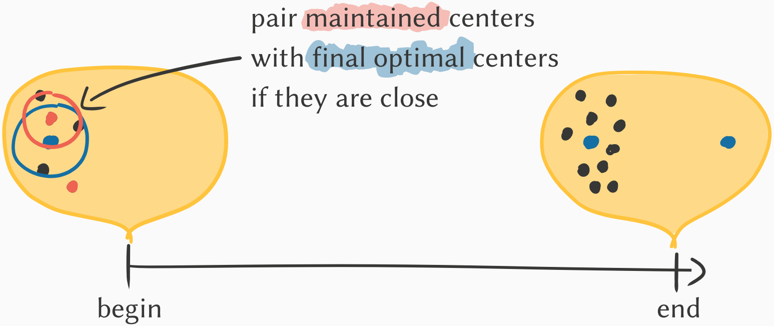
pair maintained centers
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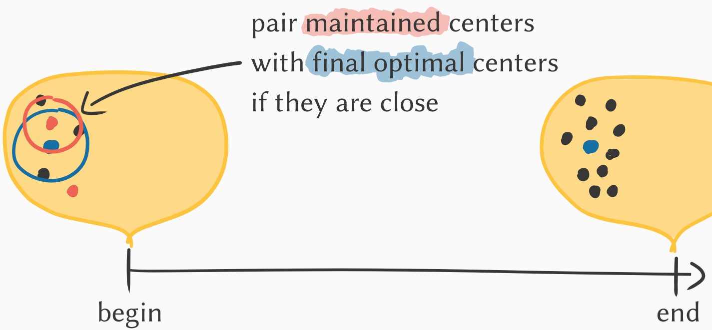
if they are close
begin
end

Algorithm



algorithm:

- remove the ℓ unpaired centers but maintain approximation
- open $\ell + 1$ inserted points as centers ($\rightarrow k + 1$ centers)
- swap $O(\ell)$ centers to obtain a good solution with k centers

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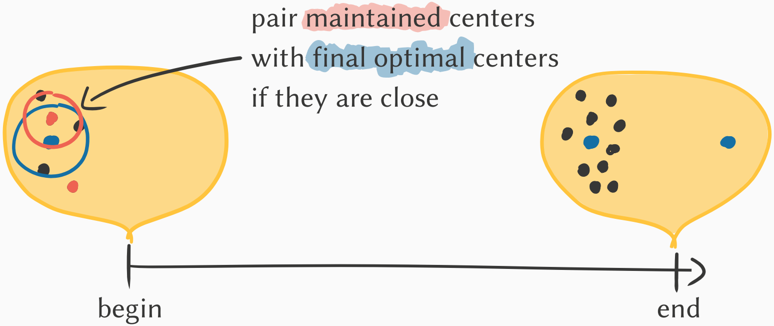

algorithm: "epoch"

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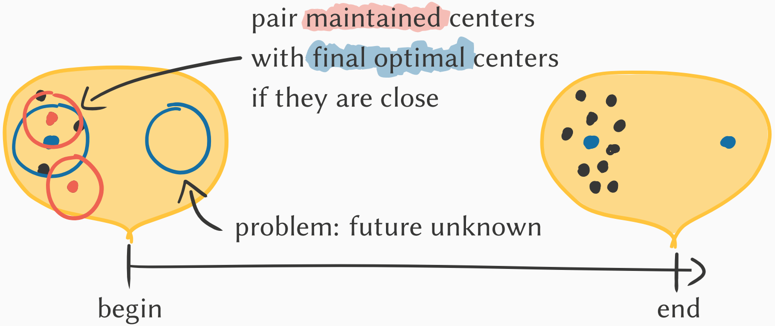

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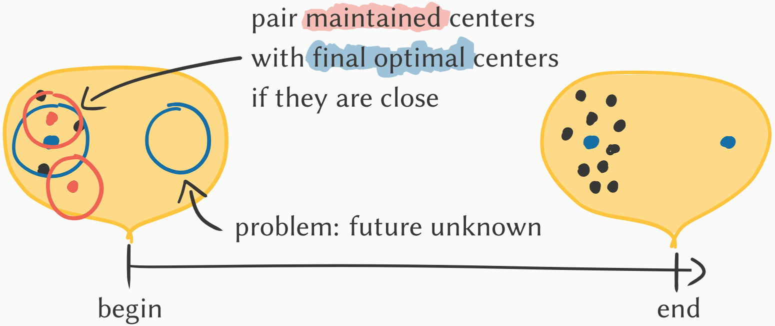

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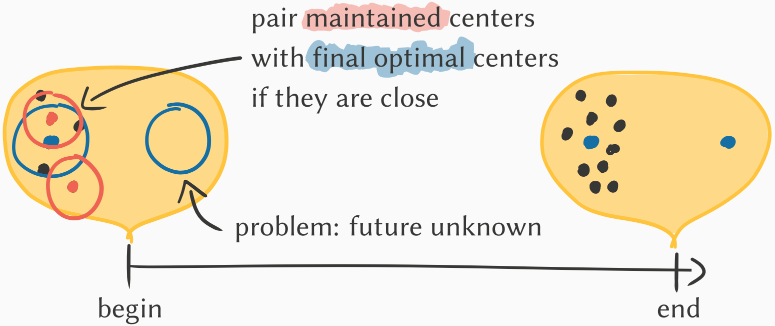

algorithm: "epoch"

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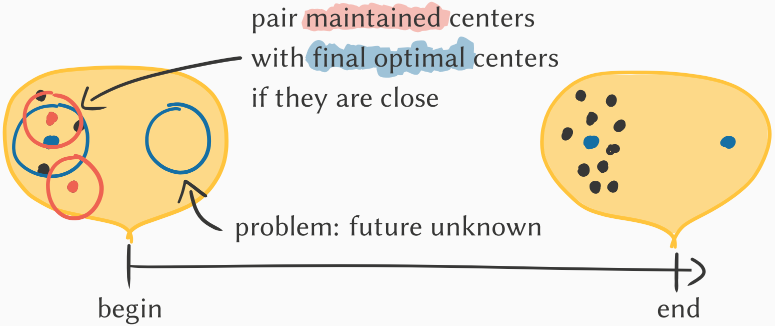

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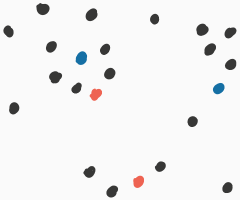
Key Elements of the Analysis

well-separated pairs



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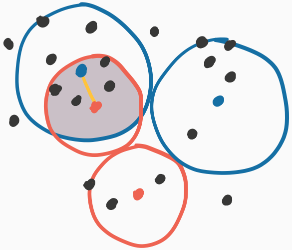

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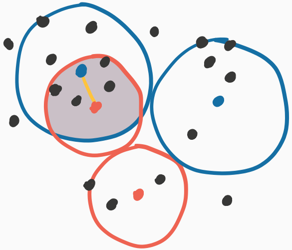

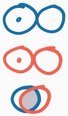

isolated optimal center and

isolated maintained center

that are close to each other

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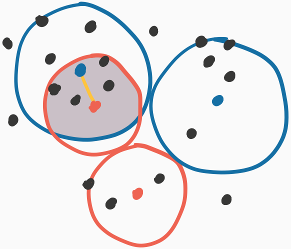

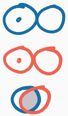

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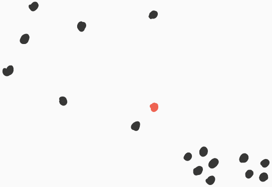

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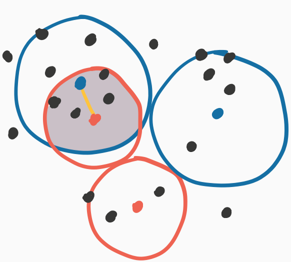

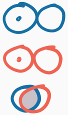

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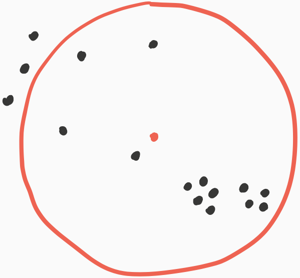

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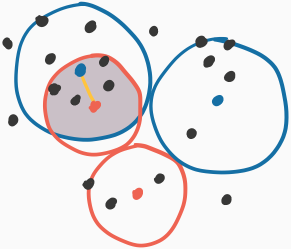

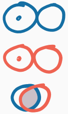

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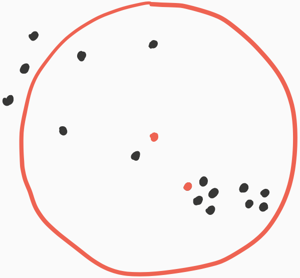

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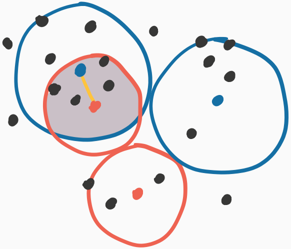

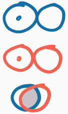

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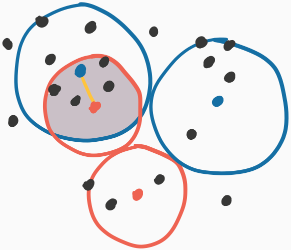

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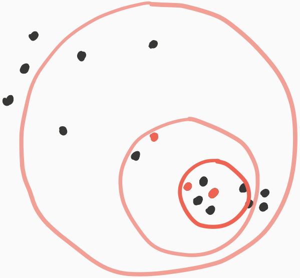

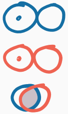

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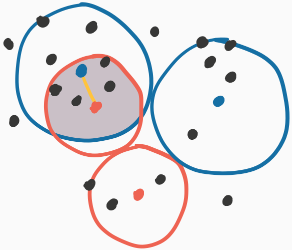

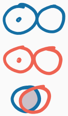

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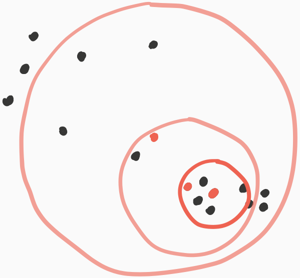

robustify each center
against future insertions
at the cluster's border

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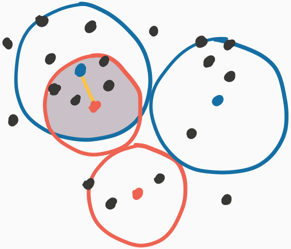

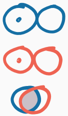

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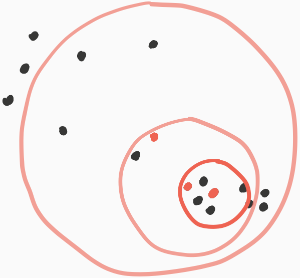

robustify each center
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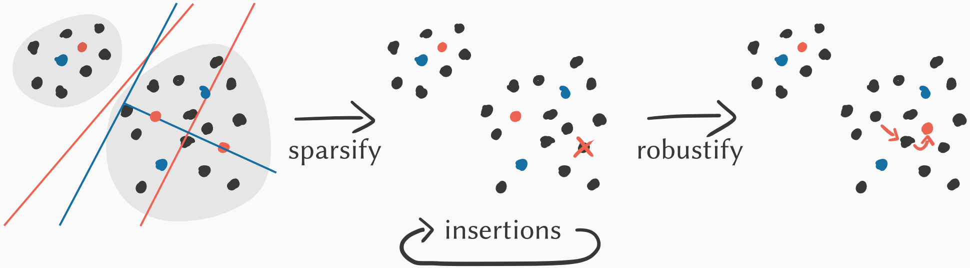
well-separated, robust centers are approximately optimal
not well-separated centers can be removed

Summary



- $O(1)$-approximate k-clustering with $O(k \cdot \text{polylog}(n, \Delta))$ consistency
- tight up to polylogarithmic factors (even for offline setting)
- analysis exploits structural properties, algorithm is based on epochs
- is there a simpler approach, e.g., by local search?