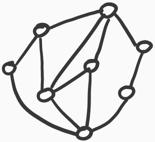
You say it’s only constant? Then something must be hyperfinite!

And I say your title is too long.

Hendrik Fichtenberger

July 20, 2019

Property Testing in a Nutshell



planar ✓

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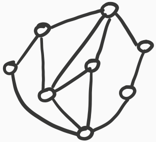

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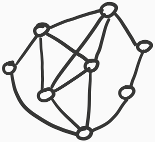

non-planar ✗

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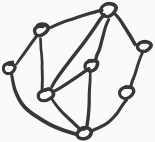

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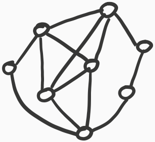

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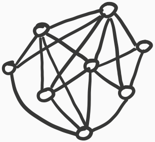

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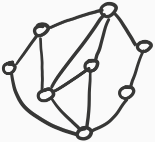

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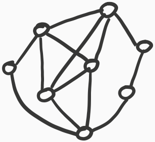

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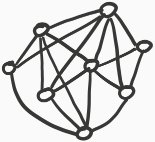

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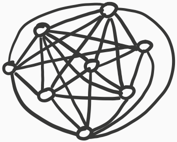

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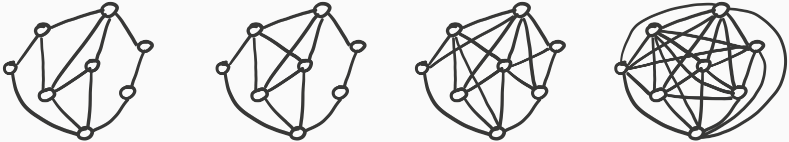
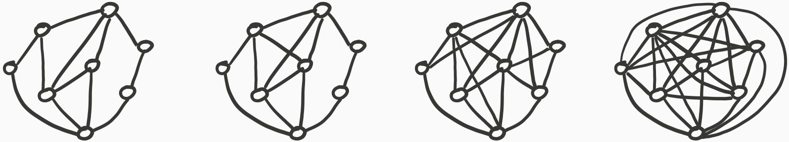


planar ✓ | non-planar ✗ | non-planar ✗ | non-planar ✗

time complexity: $\Omega(|V|)$

Property Testing in a Nutshell



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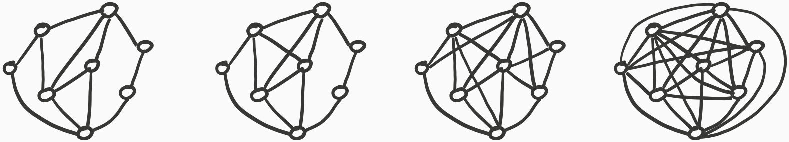

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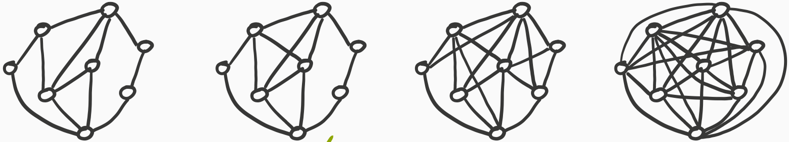

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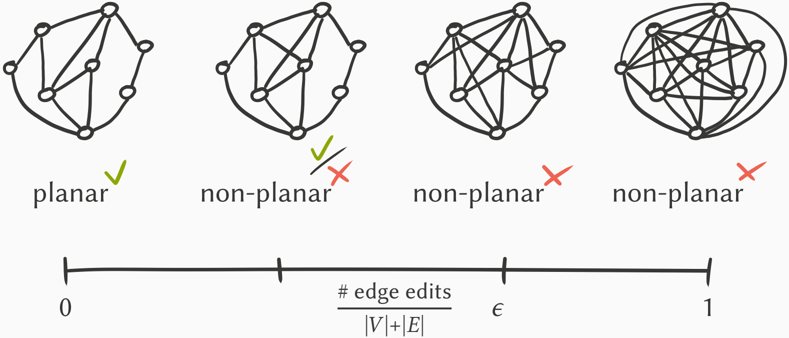

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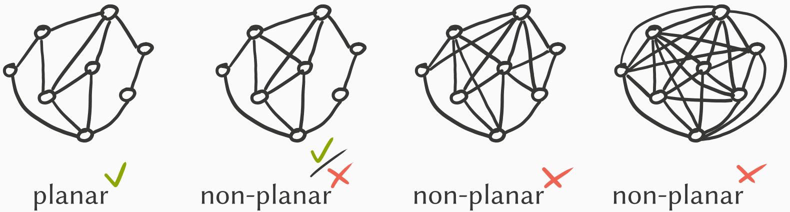

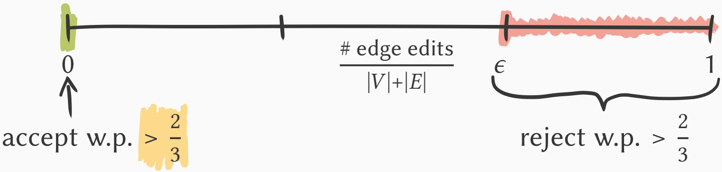

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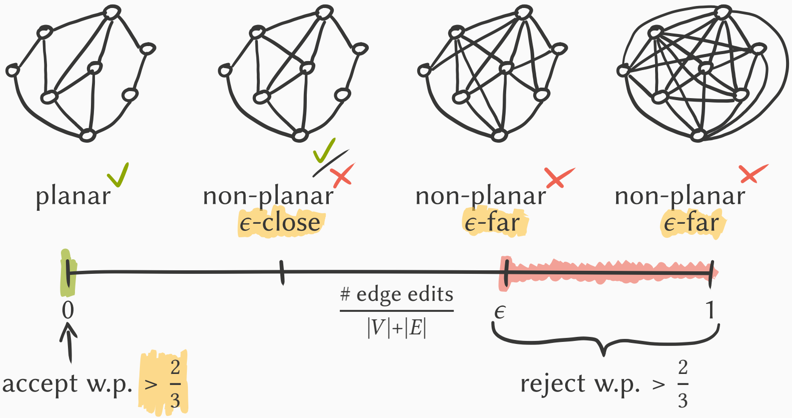

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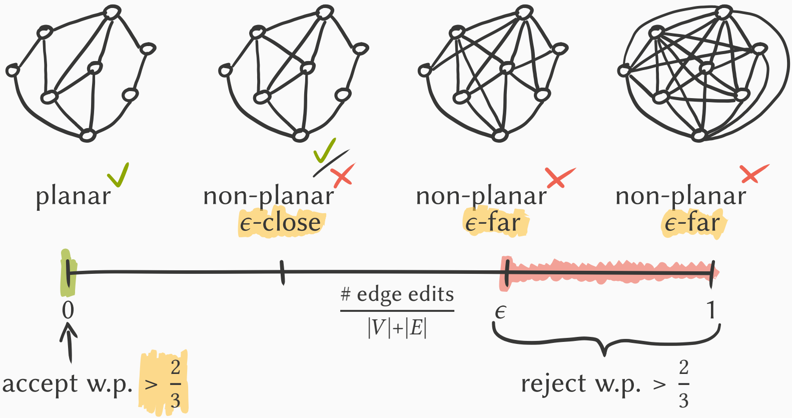
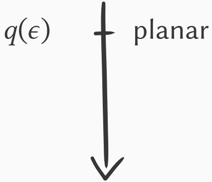


complexity: # queries to adjacency list entries

Property Testing of Bounded Degree Graphs

bounded degree graphs: $\forall v \in V \, : \, d(v) \leq d, d \in O(1)$



Property Testing of Bounded Degree Graphs

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big picture?

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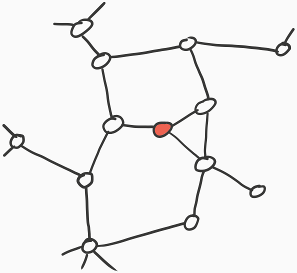
big picture?

bounded-degree graphs:
little known about
constant-time testability

$\text{disk}_k(v)$: subgraph induced by $\text{BFS}(v)$ of depth k



k-Disks and Frequency Vectors

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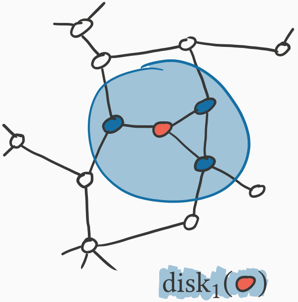

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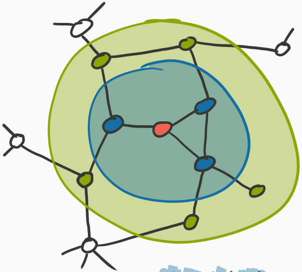

$\text{disk}_1(\bullet)$

$\text{disk}_2(\bullet)$

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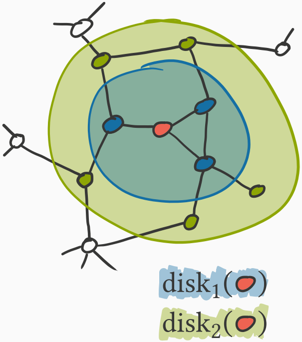
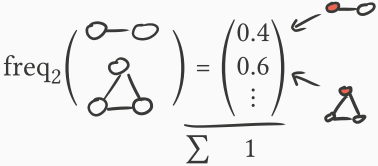


$\text{freq}_k(G)$: for each k-disk isomorphism type calculate its share of vertices



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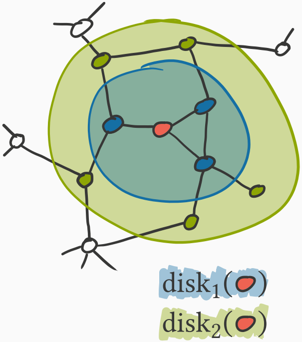
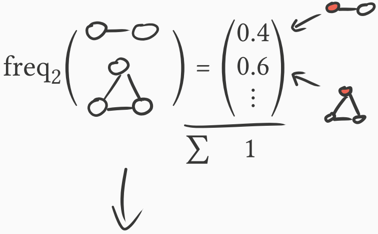
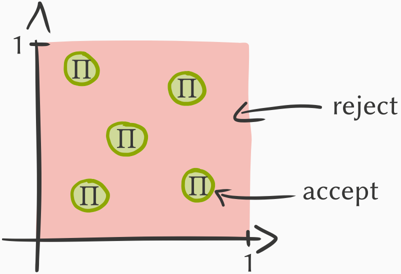




frequency vector is a locality feature

Constant-Query Testers



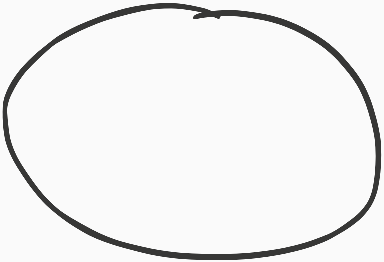
Theorem [GR'09, ...]

Every property tester with constant query complexity $q := q(\epsilon)$ can be transformed into an algorithm that

1. computes an approximation $\widetilde{\text{freq}}_{\Theta(q)}(G)$ of $\text{freq}_{\Theta(q)}(G)$
2. accepts iff $\|\widetilde{\text{freq}}_{\Theta(q)}(G) - \text{freq}_{\Theta(q)}(G')\|_1 \leq \frac{1}{\Theta(q)}$ for any $G' \in \Pi$

Hyperfinite Graphs

In every planar graph, there exists a set of $\sqrt{n}$ separators



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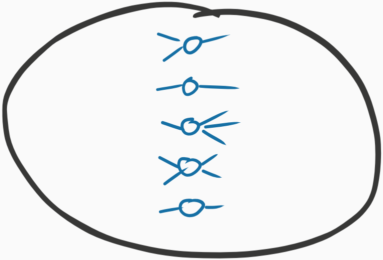

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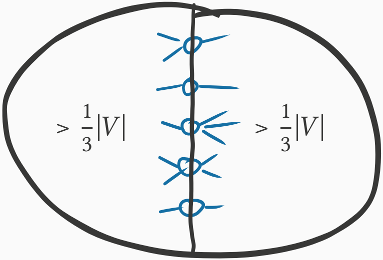

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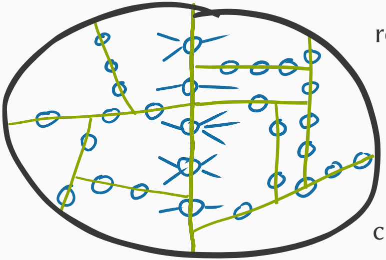
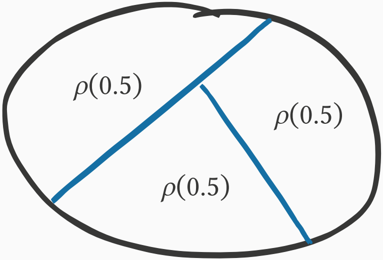


remove ϵn edges

components of size ϵ^{-2}



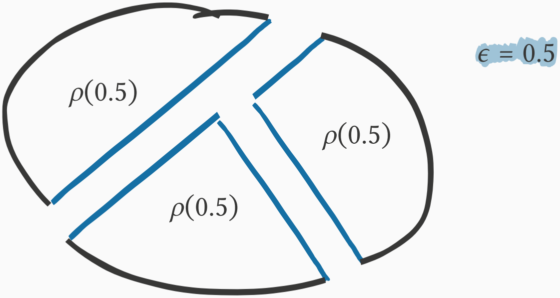
$\epsilon = 0.5$

Definition

(ϵ, s)-hyperfinite: can remove at most ϵdn edges to obtain connected components of size at most s

ρ-hyperfinite: $(\epsilon, \rho(\epsilon))$-hyperfinite for all $\epsilon \in (0, 1]$

Hyperfinite Graphs



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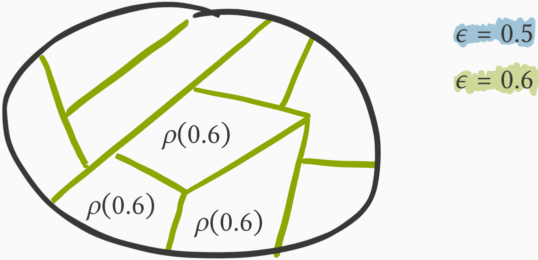

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The Story so Far

Π is characterized
by its k-disk distribution

Π is ρ-hyperfinite

Π has constant
query complexity

Newman, Sohler, STOC'11; Fichtenberger, Peng, Sohler, SODA'19

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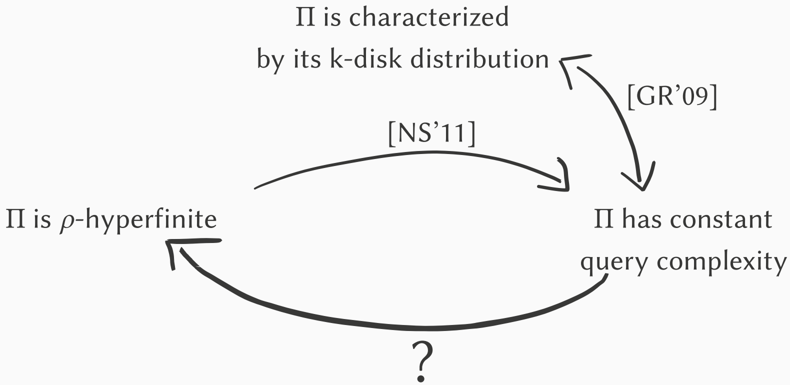

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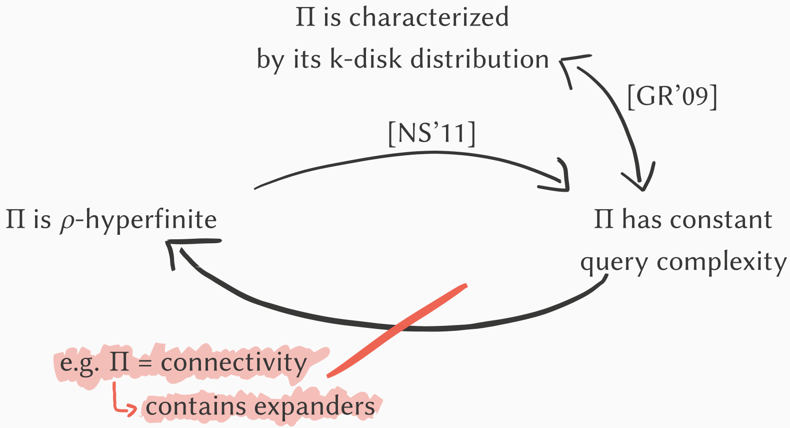

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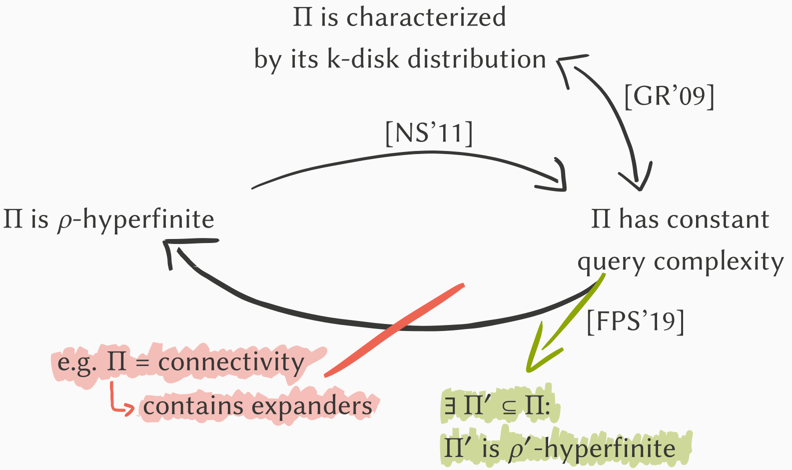

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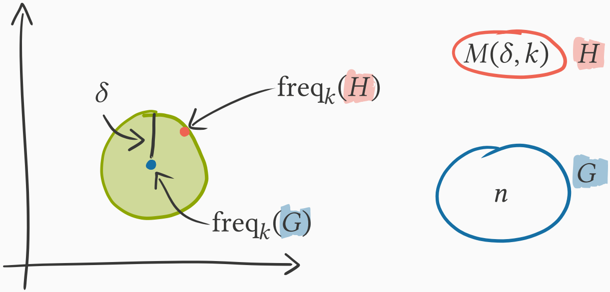
Small Frequency-Preserver Graphs



Theorem [Alon'11]

For every $\delta, k > 0$, there exists $M(\delta, k)$ such that for every G there exists H of size at most $M(\delta, k)$ and $\|\mathrm{freq}_k(G) - \mathrm{freq}_k(H)\|_1 < \delta$.

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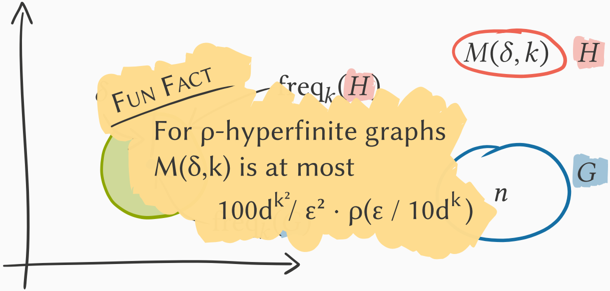

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Putting Everything Into The Property Testing Blender

start w/ $G \in \Pi$

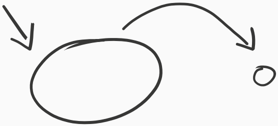
rel. Π in Π

hypf. ?

size n

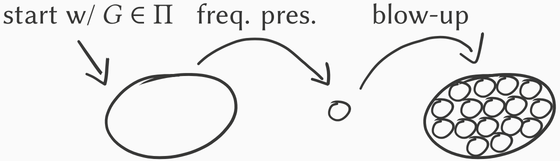
freq. v. change original

start w/ $G \in \Pi$ freq. pres.



rel. Π	in Π	?
hypf.	?	?
size	n	$O(1)$
freq. v. change	original	$< \delta/2$

Putting Everything Into The Property Testing Blender



rel. Π	in Π	?	ϵ-close to Π
hypf.	?	?	$(0, M(\delta, k))$-hyperfinite
size	n	$O(1)$	n
freq. v. change	original	$< \delta/2$	$= 0$

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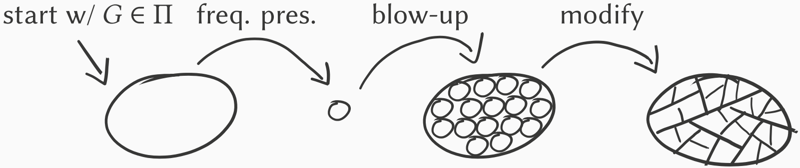
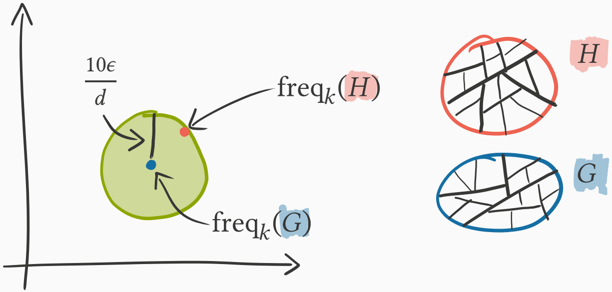


rel. Π	in Π	?	ϵ-close to Π	in Π
hypf.	?	?	$(0, M(\delta, k))$-hyperfinite	$(\epsilon, M(\delta, k))$-hyperfinite
size	n	$O(1)$	n	n
freq. v. original change		$< \delta/2$	$= 0$	$< \delta/2$

Connection Between Hyperfinite Graphs and k-Disks



Theorem [BSS'08]

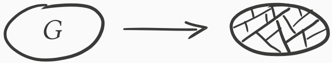
If G is $\rho(\epsilon)$-hyperfinite, then all H with $\|\mathrm{freq}_k(G) - \mathrm{freq}_k(H)\|_1 < \frac{10\epsilon}{d}$ are $\rho(f(\epsilon))$-hyperfinite for some function f.

Benjamini, Schramm, Shapira, STOC'08

...still blending...

pick $G \in \Pi$

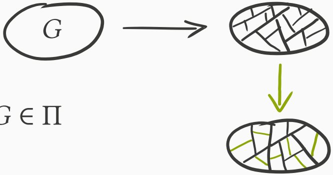
…still blending…



$(\epsilon_1, M(\delta_1, k_1))$-hyperfinite

pick $G \in \Pi$

...still blending...

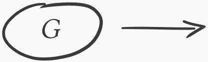


$(\epsilon_1, M(\delta_1, k_1))$-hyperfinite

pick $G \in \Pi$

$(\epsilon_2, M(\delta_2, k_2))$-hyperfinite
$(\epsilon_1', M(\delta_1, k_2))$-hyperfinite

...still blending...



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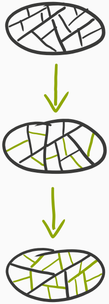

$(\epsilon_1, M(\delta_1, k_1))$-hyperfinite

$(\epsilon_2, M(\delta_2, k_2))$-hyperfinite
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$(\epsilon_3, M(\delta_3, k_3))$-hyperfinite
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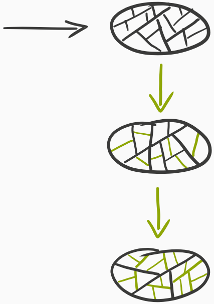
pick $G \in \Pi$



$(\epsilon_1, M(\delta_1, k_1))$-hyperfinite

$(\epsilon_2, M(\delta_2, k_2))$-hyperfinite
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$(\epsilon_3, M(\delta_3, k_3))$-hyperfinite
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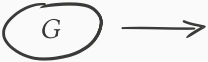
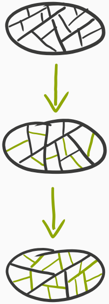
define ρ

...still blending...



for every n
pick $G \in \Pi$



$(\epsilon_1, M(\delta_1, k_1))$-hyperfinite

$(\epsilon_2, M(\delta_2, k_2))$-hyperfinite
$(\epsilon_1', M(\delta_1, k_2))$-hyperfinite

$(\epsilon_3, M(\delta_3, k_3))$-hyperfinite
$(\epsilon_2', M(\delta_2, k_2))$-hyperfinite
$(\epsilon_1', M(\delta_1, k_2))$-hyperfinite

define ρ

Open (Bl)ending